UDC 539.3

ON A METHOD OF SOLVING DUAL INTEGRAL EQUATIONS

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We expound a method of reducing a class of dual integral equations which find important practical application to infinite algebraic systems of the first kind. The latter system can be reduced to the systems of the second kind by exact inversion of the principal singular part, and the second kind systems can be solved using the method of consecutive approximations [1 – 6]. The dual integral equations generated by the Kontorovich-Lebedev and Mehler-Fock integral transforms are considered as examples as well as the problems of torsion of a truncated elastic sphere by a punch and that of a circular crack in an elastic space.

1. General theory. Let the following second order linear differential equation in x be given:

$$(L - u^2)\, y = 0, \quad 0 \leqslant x < \infty \tag{1.1}$$

By solving the corresponding Sturm-Liouville problem for this equation on the interval $x \in [0, \infty)$, we construct the following integral transform:

$$g(x) = \int_0^\infty G(u)\, B(u, x)\, du, \quad G(u) = \int_0^\infty g(\xi)\, M(u, \xi)\, d\xi \tag{1.2}$$

Here $B(u, x)$ is the eigenfunction of (1.1) for any $u \in [0, \infty)$, vanishing at infinity and bounded at zero. We shall call this function a solution of the first kind.

Let us now consider the dual integral equation

$$\int_0^\infty Q(u)\, K(u)\, B(u, x)\, du = f(x), \quad a \leqslant x \leqslant b \tag{1.3}$$

$$\int_0^\infty Q(u)\, B(u, x)\, du = 0, \quad 0 < x < a, \quad b < x < \infty$$

$$K(u) = A\, \frac{P_1(u^2)}{P_2(u^2)} = A \prod_{n=0}^{\infty} \left(1 + \frac{u^2}{\delta_n{}^2}\right)\left(1 + \frac{u^2}{\gamma_n{}^2}\right)^{-1}, \quad A = \mathrm{const} \tag{1.4}$$

Here $K(u)$ is an even meromorphic function of the form (1.4) where $i\delta_n$ and $i\gamma_n$ represent a denumerable set of zeros and poles of $K(u)$. We assume that there are no multiple zeros and poles and that $\delta_n \neq \gamma_m$ $(n, m = 1, 2, 3 \ldots)$. Let also δ_n and γ_n increase monotonously in absolute value with increasing n thus ensuring the convergence of the infinite product (1.4). Let also the following estimate hold on any proper system of contours C_n $(C_n \subset C_{n+1})$:

$$K(u) = O(|u|^p), \quad p \leqslant 1 \tag{1.5}$$

Utilizing the fact that by applying the operational L to the function $B(u, x)$ we obtain $u^2 B(u, x)$, and taking into account (1.4), we can write the first relation of the dual equation (1.3) in the form

$$AP_1(L)\,q(x) = P_2(L)\,f(x), \quad a \leqslant x \leqslant b \tag{1.6}$$

$$q(x) = \int_0^\infty Q(u)\,B(u, x)\,du \tag{1.7}$$

where $P_1(L)$ and $P_2(L)$ are infinite order differential operators in x.

The solution of (1.6) with respect to $q(x)$ can be written in the form

$$q(x) = \frac{P_2(L)}{AP_1(L)} f(x) + \sum_{n=1}^\infty [C_n B(i\delta_n, x) + D_n R(i\delta_n, x)] \tag{1.8}$$
$$a \leqslant x \leqslant b$$

Here the first term represents a particular solution of an inhomogeneous equation obtained by symbolic methods, and the infinite sum gives the general solution of the homogeneous equation. The function $R(u, x)$ represents a solution of the second kind of the equation (1.1), and is linearly independent of $B(u, x)$.

From (1.7), (1.8) and the second relation of (1.3) we obtain

$$Q(u) = \int_a^b q(\xi)\,M(u, \xi)\,d\xi \tag{1.9}$$

In the following we shall set $f(x) = B(\varepsilon, x)$, remembering that the function $f(x)$ can, in a general case, be represented by the integral (1.2) or approximated by means of a linear combination of the function $B(\varepsilon_k, x)$ $(k = 1,2,3, \ldots, N)$. The first term in (1.8) now assumes the form $K^{-1}(\varepsilon)\,B(\varepsilon, x)$. Taking this into account and substituting (1.8) into (1.9), we obtain

$$Q(u) = K^{-1}(\varepsilon)\,\varphi(u, -i\varepsilon) + \sum_{n=1}^\infty [C_n \varphi(u, \delta_n) + D_n \psi(u, \delta_n)] \tag{1.10}$$

$$\varphi(u, x) = \int_a^b B(ix, \xi)\,M(u, \xi)\,d\xi, \quad \psi(u, x) = \int_a^b R(ix, \xi)\,M(u, \xi)\,d\xi$$

The constants C_n and D_n must now be found from the condition that the solution (1.10) satisfies the initial dual equation (1.3).

We note that under the assumptions made above, we can represent the meromorphic function $K(u)$ by a sum of its principal values

$$K(u) = A - \frac{2}{\pi} \sum_{k=1}^\infty \frac{s_k u^2}{\gamma_k (u^2 + \gamma_k^2)}, \quad s_k = \pi i \{[K^{-1}(i\gamma_k)]'\}^{-1} \tag{1.11}$$

$$\frac{2}{\pi} \sum_{k=1}^\infty \frac{s_k}{\gamma_k} = A \quad \text{for } p < 0$$

The relation (1.11) yields expressions for $K(u) - K(\varepsilon)$ and for $K(u) - K(i\delta_n)$. Inserting now (1.10) into the first relation of (1.3) and taking into account the fact that

$$\int_0^\infty \varphi(u, -i\varepsilon)\,B(u, x)\,du = f(x) = B(\varepsilon, x), \quad a \leqslant x \leqslant b$$

by definition, and also that $K\,(i\delta_n) = 0$, we obtain

$$K^{-1}(\varepsilon)\int_0^\infty \varphi(u, -i\varepsilon)\,[K(u)-K(\varepsilon)]\,B(u,x)\,du +$$

$$\sum_{n=1}^{\infty}\Big\{C_n\int_0^\infty \varphi(u,\delta_n)\,[K(u)-K(i\delta_n)]\,B(u,x)\,du +$$

$$D_n\int_0^\infty \psi(u,\delta_n)\,[K(u)-K(i\delta_n)]\,B(u,x)\,du\Big\} = 0$$

Taking also into account the expressions for the differences within the square brackets, we obtain

$$\sum_{k=1}^{\infty} s_k\gamma_k\left\{\frac{K^{-1}(\varepsilon)}{\gamma_k^2+\varepsilon^2}T_k(x,-i\varepsilon) + \sum_{n=1}^{\infty}\frac{1}{\gamma_k^2-\delta_n^2}[C_nT_k(x,\delta_n) + \qquad (1.12)$$

$$D_nU_k(x,\delta_n)]\right\} = 0, \quad a \leqslant x \leqslant b$$

$$T_k(x,\mu) = \int_0^\infty \frac{u^2+\mu^2}{u^2+\gamma_k^2}\varphi(u,\mu)\,B(u,x)\,du$$

$$U_k(x,\mu) = \int_0^\infty \frac{u^2+\mu^2}{u^2+\gamma_k^2}\psi(u,\mu)\,B(u,x)\,du$$

Futher study of the relations (1.12) cannot be made without knowing the particular forms assumed by the functions $B\,(u,\,x)$, $\varphi\,(u,\,x)$ and $\psi\,(u,\,x)$. Nevertheless we can establish the final structure of (1.12). To do this we solve the differential equation (1.6) with respect to the function $f\,(x)$. We have

$$f(x) = \frac{AP_1(L)}{P_2(L)}q(x) + \sum_{k=1}^{\infty}[E_kB(i\gamma_k,x) + F_kR(i\gamma_k,x)], \quad a\leqslant x\leqslant b \qquad (1.13)$$

The constants E_k and F_k can be found from the condition that (1.13) is equivalent to the first relation of (1.3) and hence to (1.12). Obviously, E_k and F_k are linear functionals in $q\,(x)$, i.e. $E_k = E_k\,(q)$ and $F_k = F_k\,(q)$. Let us now substitute into (1.13) the expression (1.8) for $q\,(x)$. Taking also into account the fact that

$$\frac{AP_1(L)}{P_2(L)}B(i\delta_n,x) = K(i\delta_n)\,B(i\delta_n,x) = 0$$

$$\frac{AP_1(L)}{P_2(L)}R(i\delta_n,x) = K(i\delta_n)\,R(i\delta_n,x) = 0$$

we reduce (1.13),and hence (1.12), to the form

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}[(C_nE_{kn}^+ + D_nF_{kn}^+)\,B(i\gamma_k,x) + (C_nE_{kn}^- +$$

$$D_nF_{kn}^-)\,R(i\gamma_k,x)] + K^{-1}(\varepsilon)\sum_{k=1}[E_k^*B(i\gamma_k,x) + F_k^*R(i\gamma_k,x)] = 0$$

$$E_k[B(\varepsilon,x)] = E_k^*, \qquad E_k[B(i\delta_n,x)] = E_{kn}^+, \qquad E_k[R(i\delta_n,x)] = E_{kn}^-$$

$$F_k[B(\varepsilon,x)] = F_k^*, \qquad F_k[B(i\delta_n,x)] = F_{kn}^+, \qquad F_k[R(i\delta_n,x)] = F_{kn}^-$$

Assuming finally that the functions $B\ (i\gamma_k,\ x)$ and $R\ (i\gamma_k,\ x)$ $(k = 1,\ 2,\ 3\ \ldots)$ are linearly independent, we obtain the following two infinite algebraic systems of the first kind for the constants C_n and D_n entering the solution (1.10) of the dual equation (1.3):

$$K^{-1}(\varepsilon)\,E_k^* + \sum_{n=1}^{\infty}(C_n E_{kn}^+ + D_n F_{kn}^+) = 0 \qquad (k = 1, 2, 3, \ldots) \tag{1.14}$$

$$K^{-1}(\varepsilon)\,F_k^* + \sum_{n=1}^{\infty}(C_n E_{kn}^- + D_n F_{kn}^-) = 0$$

When considering particular examples, we shall construct the systems (1.14) by direct computation of the integrals in (1.12).

2. Dual integral equations generated by the Kontorovich-Lebedev and Mehler-Fock integral transforms. If we assume that in (1.1)

$$L = x^2 - x^2\frac{d^2}{dx^2} - x\frac{d}{dx}, \qquad 0 < x < \infty \tag{2.1}$$

then the functions $B\ (u,\ x)$ and $M\ (u,\ \xi)$ in the integral transform (1.2) have the form

$$B\ (u,\ x) = K_{iu}(x), \quad M\ (u,\ \xi) = 2u\pi^{-2}\operatorname{sh}\pi u\,\xi^{-1}\,K_{iu}(\xi) \tag{2.2}$$

where $K_{iu}(x)$ is the MacDonald's function. The transform (1.2), (2.2) obtained here is the Kontorovich-Lebedev integral transform. The dual integral equation (1.3) generated by this transform can, in accordance with Sect. 1, be reduced to (1.12) in which

$$\varphi(u, x) = \frac{2u\operatorname{sh}\pi u}{\pi^2}\int_a^b K_{-x}(\xi)\,K_{iu}(\xi)\frac{d\xi}{\xi} \tag{2.3}$$

$$\psi(u, x) = \frac{2u\operatorname{sh}\pi u}{\pi^2}\int_a^b I_{-x}(\xi)\,K_{iu}(\xi)\frac{d\xi}{\xi}$$

where $(I_{-x}(\xi) = R\ (ix,\ \xi)$ is a cylindrical function of an imaginary argument, and the solution of this dual integral equation can be written in the form of (1.10).

Let us transform the relations (1.12) into an infinite system of linear algebraic equations with respect to the unknown C_n and D_n given in the expansion (1.8), by computing the integrals in (2.3) and (1.12). The integrals in (2.3) can be found with the help of the well known relations given in [7], while the computation of the integrals in (1.12) can be reduced to finding an integral of the form

$$S\ (x, d) = \int_0^\infty \frac{\tau\operatorname{sh}\pi\tau}{\tau^2 + \gamma_k^2}\,K_{i\tau}(x)\,K_{i\tau}(d)\,d\tau \tag{2.4}$$

Using the integral transforms

$$K_{i\tau}(x) = \operatorname{ch}^{-1}\frac{\pi\tau}{2}\int_0^\infty \cos(x\operatorname{ch}s)\cos\tau s\,ds$$

$$K_{i\tau}(x) = \operatorname{ch}^{-1}\frac{\pi\tau}{2}\int_0^\infty \sin(x\operatorname{sh}y)\sin\tau y\,dy$$

$$I_{i\tau}(x) = \frac{2}{\pi}\,\mathrm{ch}\,\frac{\pi\tau}{2}\int_0^\infty \sin(x\,\mathrm{sh}\,y)\,e^{-i\tau y}dy$$

we compute the integral (2.4) for any γ_k

$$S(x,d) = \begin{cases} {}^1/_2\pi^2 I_{\gamma_k}(x)\,K_{\gamma_k}(d), & 0 < x < a \\ {}^1/_2\pi^2 K_{\gamma_k}(x)\,I_{\gamma_k}(d), & d < x < \infty \end{cases}$$

For $\gamma_k = k\,(k = 1, 2, 3, \ldots)$, the above expression agrees with a well known result (see [8]).

Substituting values of the functions $T_k(x, \mu)$ and $U_k(x, \mu)$ thus obtained from (1.12) into the first relation of (1.12) and equating to zero the coefficients accompanying the linearly independent functions $K_{\gamma_k}(x)$ and $I_{\gamma_k}(x)$, we obtain the following infinite system for determining the unknowns C_n and D_n:

$$\frac{K^{-1}(\varepsilon)}{\varepsilon^2+\gamma_k^2}\left[K_{i\varepsilon}'(b)\,K_{\gamma_k}(b) - K_{i\varepsilon}(b)\,K_{\gamma_k}'(b)\right] +$$

$$\sum_{n=1}^{\infty}\frac{C_n}{\delta_n^2-\gamma_k^2}\left[K_{-\delta_n}'(b)\,K_{\gamma_k}(b) - K_{-\delta_n}(b)\,K_{\gamma_k}'(b)\right] +$$

$$\sum_{n=1}^{\infty}\frac{D_n}{\delta_n^2-\gamma_k^2}\left[I_{-\delta_n}'(b)\,K_{\gamma_k}(b) - I_{-\delta_n}(b)\,K_{\gamma_k}(b)\right] = 0$$

$$\frac{K^{-1}(\varepsilon)}{\varepsilon^2+\gamma_k^2}\left[K_{i\varepsilon}'(a)\,I_{\gamma_k}(a) - K_{i\varepsilon}(a)\,I_{\gamma_k}'(a)\right] +$$

$$\sum_{n=1}^{\infty}\frac{C_n}{\delta_n^2-\gamma_k^2}\left[K_{-\delta_n}'(a)\,I_{\gamma_k}(a) - K_{-\delta_n}(a)\,I_{\gamma_k}'(a)\right] +$$

$$\sum_{n=1}^{\infty}\frac{D_n}{\delta_n^2-\gamma_k^2}\left[I_{-\delta_n}'(a)\,I_{\gamma_k}(a) - I_{-\delta_n}(a)\,I_{\gamma_k}'(a)\right] = 0, \quad k = 1, 2, 3, \ldots$$

An analogous infinite system was obtained in [6] by a different method, and was reduced to an infinite system of the second kind. A zeroth approximation was also constructed for it, with $|\delta_n| \to \infty$ and $|\gamma_k| \to \infty$.

If we now assume that in (1.1)

$$L = (1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} - \frac{m^2}{1-y^2} - \frac{1}{4}, \quad y = \mathrm{ch}\,x,\ 0 \leqslant x < \infty$$

then the functions $B(u, x)$ and $M(u, \xi)$ in the integral transform (1.2) become

$$B(u,x) = P^m_{-1/2+iu}(\mathrm{ch}\,x), \qquad M(u,\xi) = u\,\mathrm{th}\,\pi u\,P^{-m}_{-1/2+iu}(\mathrm{ch}\,\xi) \tag{2.5}$$

where $P^m_{-1/2+iu}(\mathrm{ch}\,x)$ is the Legendre function. The transform (1.2), (2.5) obtained here is the Mehler-Fock integral transform. The dual integral equation (1.3) generated by this transform for $a = 0$, becomes:

$$\int_0^\infty Q(\tau)\,K(\tau)\,P^m_{-1/2+i\tau}(\mathrm{ch}\,x)\,d\tau = f(x), \quad 0 \leqslant x \leqslant b \tag{2.6}$$

$$\int_0^\infty Q(\tau)\,P^m_{-1/2+i\tau}(\mathrm{ch}\,x)\,d\tau = 0, \qquad b < x < \infty$$

In accordance with Sect. 1, its solution can be constructed in the form

$$Q(u) = u \operatorname{th} \pi u \int_0^b q(\xi) P^{-m}_{-1/2+iu}(\operatorname{ch}\xi) \operatorname{sh}\xi d\xi \tag{2.7}$$

$$q(x) = \frac{P_2(L)}{AP_1(L)} f(x) + \sum_{n=1}^{\infty} [C_n P^m_{-1/2-\delta n}(\operatorname{ch} x) + D_n Q^m_{-1/2-\delta n}(\operatorname{ch} x)]$$

$$0 \leqslant x \leqslant b$$

Imposing on the function $q(x)$ the conditions of boundedness at $x = 0$ and taking into account the fact that $Q^m_{1/2-\delta n}(\operatorname{ch} x)$ has a logarithmic singularity at $x = 0$, we set $D_n = 0$ $(n = 1, 2, 3, \ldots)$. In the following we shall also assume $m = 0$, since when $m \neq 0$, the dual equation (2.6) can be reduced [9] using the relation

$$P^m_{-1/2+i\tau}(x) = (x^2-1)^{m/2} \frac{d^m P_{-1/2+i\tau}(x)}{dx^m} \tag{2.8}$$

to the dual equation (2.6) with $m = 0$.

Let us now transform the expression (1.12) into an infinite algebraic system in the unknown C_n from (2.7), by finding $\varphi(u, x)$ from (1.10) and $T_k(x, \mu)$ from (1.12). We have

$$\varphi(u, x) = u \operatorname{th} \pi u \int_0^b P_{-1/2-x}(\operatorname{ch} y) P_{-1/2+iu}(\operatorname{ch} y) \operatorname{sh} y \, dy = \tag{2.9}$$

$$\frac{\operatorname{sh} b}{x^2+u^2}[P_{-1/2+iu}(\operatorname{ch} b) P^1_{-1/2-x}(\operatorname{ch} b) -$$

$$P_{-1/2-x}(\operatorname{ch} b) P^1_{-1/2+iu}(\operatorname{ch} b)] u \operatorname{th} \pi u$$

Using (2.9) we can reduce the problem of finding the function $T_k(x, \mu)$ to that of computing integrals of the form

$$S_1(x, b) = \int_0^\infty \frac{\tau \operatorname{th} \pi\tau}{\tau^2 + \gamma_k^2} P_{-1/2+i\tau}(\operatorname{ch} x) P_{-1/2+i\tau}(\operatorname{ch} b) d\tau$$

The latter can be solved using the integral representations [9] of the functions $P_{-1/2+i\tau}(\operatorname{ch} x)$ and $Q_{-1/2+i\tau}(\operatorname{ch} b)$. We have

$$S_1(x, b) = \begin{cases} P_{-1/2-\gamma_k}(\operatorname{ch} x) Q_{-1/2+\gamma_k}(\operatorname{ch} b), & x < b \\ Q_{-1/2+\gamma_k}(\operatorname{ch} x) P_{-1/2-\gamma_k}(\operatorname{ch} b), & b < x \end{cases}$$

Substituting the function $T_k(x, \mu)$ obtained in this manner into the first relation of (1.12) and equating to zero the coefficients accompanying the linearly independent functions $P_{-1/2-\gamma_k}(\operatorname{ch} x)$ $(k = 1, 2, 3, \ldots)$, we obtain the following infinite system for the unknown C_n:

$$\frac{K^{-1}(\varepsilon)}{\varepsilon^2+\gamma_k^2}[P_{-1/2+i\varepsilon}(\operatorname{ch} b) Q^1_{-1/2+\gamma_k}(\operatorname{ch} b) - P^1_{-1/2+i\varepsilon}(\operatorname{ch} b) Q_{-1/2+\gamma_k}(\operatorname{ch} b)] +$$

$$\sum_{n=1}^{\infty} \frac{C_n}{\delta_n^2 - \gamma_k^2}[P^1_{-1/2-\delta_n}(\operatorname{ch} b) Q_{-1/2+\gamma_k}(\operatorname{ch} b) -$$

$$P_{-1/2-\delta_n}(\operatorname{ch} b) Q^1_{-1/2+\gamma_k}(\operatorname{ch} b)] = 0, \quad k = 1, 2, 3, \ldots$$

Let us introduce into this system the following new unknowns:

$$x_n = -i\sqrt{\pi} C_n [\sqrt{2 \operatorname{ch} b}\, \delta_n Q_{-1/2+\delta_n}(\operatorname{ch} b)]^{-1}$$

The system then becomes

$$BX = D, \quad X = \| x_n \| \tag{2.10}$$

$$B = \| b_{kn} \| = \left\| \frac{i\delta_n \Gamma(\gamma_k + 1)\, Q_{-1/2+\delta_n}(\operatorname{ch} b)\, \pi^{-1/2}}{(\delta_n^2 - \gamma_k^2)\, \Gamma(\gamma_k + 1/2)\, (2 \operatorname{ch} b)^{-\gamma_k - 3/2}} [P^1_{-1/2-\delta_n}(\operatorname{ch} b) \times Q_{-1/2+\gamma_k}(\operatorname{ch} b) - P_{-1/2-\delta_n}(\operatorname{ch} b)\, Q^1_{-1/2+\gamma_k}(\operatorname{ch} b)] \right\|$$

$$D = \| d_k \| = \left\| \frac{-K^{-1}(\varepsilon)\, \Gamma(\gamma_k + 1)\, (2 \operatorname{ch} b)^{\gamma_k + 1}}{(\varepsilon^2 + \gamma_k^2)\, \Gamma(\gamma_k + 1/2)} [P_{-1/2+i\varepsilon}(\operatorname{ch} b) \times Q^1_{-1/2+\gamma_k}(\operatorname{ch} b) - P^1_{-1/2+\gamma_k}(\operatorname{ch} b)\, Q_{-1/2+\gamma_k}(\operatorname{ch} b)] \right\|$$

It can be verified that for $b \to \infty$, the matrix B tends to the matrix

$$A = \| (i\gamma_k - i\delta_n)^{-1} \|$$

Using the inverse matrix A^{-1} [1] we can reduce (2.10) to the form

$$X = A^{-1}(A - B)\, X + A^{-1} D \tag{2.11}$$

and the solution of (2.11) can be constructed by the method of consecutive approximations for large values of b. For $b \to \infty$ the system (2.10) becomes

$$AX = D^\circ, \quad D^\circ = \lim_{b \to \infty} D = \left\| \frac{K^{-1}(\varepsilon)\, \Gamma(i\varepsilon)\, (2 \operatorname{ch} b)^{i\varepsilon}}{(\gamma_k - i\varepsilon)\, \Gamma(i\varepsilon + 1/2)} \right\|$$

and its solution [1] is

$$x_n = \frac{i\Gamma(i\varepsilon)\, (2 \operatorname{ch} b)^{i\varepsilon}}{K_+(-\varepsilon)\, \Gamma(i\varepsilon + 1/2)\, (i\delta_n + \varepsilon)\, K_+'(-i\delta_n)}, \quad n = 1, 2, 3, \ldots \tag{2.12}$$

$$K(u) = K_+(u)\, K_-(u)$$

where $K_+(u)$ and $K_-(u)$ are functions regular in the upper and lower semi-plane, respectively. The expression (2.12) obviously represents the principal term of the asymptotics of (2.10) as $b \to \infty$. When $b \to \infty$, the second expression of (2.7) assumes the form

$$q(x) = K^{-1}(\varepsilon)\, P_{-1/2+i\varepsilon}(\operatorname{ch} x) - \frac{\Gamma(i\varepsilon)\, (2 \operatorname{ch} b)^{i\varepsilon + 1/2}}{\sqrt{\pi} K_+(-\varepsilon)\, \Gamma(i\varepsilon + 1/2)} \times \sum_{n=1}^{\infty} \frac{\delta_n Q_{-1/2+\delta_n}(\operatorname{ch} b)\, P_{-1/2-\delta_n}(\operatorname{ch} x)}{(i\delta_n + \varepsilon)\, K_+'(-i\delta_n)}, \quad 0 \leqslant x \leqslant b \tag{2.13}$$

3. Examples. Let us set in the dual integral equation (2.6)

$$K(\tau) = \tau^{-1} \operatorname{th} \gamma\tau, \quad f(x) = P_{-1/2+i\varepsilon}(\operatorname{ch} x), \quad m = 0$$

$$\varepsilon = -i(l + 1/2), \quad b = 1, 2, \ldots$$

Then

$$\delta_n = \pi n / \gamma, \qquad \gamma_k = \pi(2k + 1)/2\gamma$$

$$K_+'(-i\delta_n) = i\sqrt{\pi} \left(\frac{\gamma}{\pi} \right)^{1/2} \frac{2^n (n-1)!}{(2n-1)!!}$$

Using the asymptotics of the Legendre functions for large value of their argument it is easy to confirm that the series in (2.13) converges for $0 \leqslant x < b$ and diverges for $x = b$. Moreover, (2.13) can be transformed into another expression containing an explicitly separated singularity, which is important for practical applications. We have

$$q_l(x) = K^{-1}\left[-i\left(l+\frac{1}{2}\right)\right] P_l(\operatorname{ch} x) + \Gamma\left(l+\frac{1}{2}\right)(2\operatorname{ch} b)^{l+1} \times \tag{3.1}$$
$$\Gamma\left[1+\frac{\gamma}{\pi}\left(l+\frac{1}{2}\right)\right]\left\{2\gamma\Gamma(l+1)\Gamma\left[\frac{1}{2}+\frac{\gamma}{\pi}\left(l+\frac{1}{2}\right)\right]\sqrt{\operatorname{ch} b \operatorname{ch} x}\right\}^{-1} \times$$
$$\left\{\left[1-\left(\frac{\operatorname{ch} x}{\operatorname{ch} b}\right)^{\pi/\gamma}\right]^{-1/2} + \frac{\gamma(2l+1)}{2\pi}\sum_{n=1}^{\infty}\left[n-\frac{\gamma(2l+1)}{2\pi}\right]^{-1} \times\right.$$
$$\left.\frac{(2n-1)!!}{(2n)!!}\left(\frac{\operatorname{ch} x}{\operatorname{ch} b}\right)^{\pi n/\gamma}\right\}$$

The infinite sum in (3.1) now converges for $0 \leqslant x \leqslant b$.

1°. We consider the mixed problem of the theory of elasticity on extension of an elastic space with a plane annular slit whose internal and external diameter is denoted by c and d, respectively, acted upon at infinity by the forces p perpendicular to the plane of the slit.

The above problem [10] can be reduced to the dual integral equation (2.6) in which (r denotes the distance from the axis of symmetry)

$$K(\tau) = \tau^{-1}\operatorname{th}\pi\tau, \quad f(x) = (1+\operatorname{ch} x)^{-1}, \operatorname{ch} x = \frac{d^2+r^2}{d^2-r^2}, \ \operatorname{ch} b = \frac{d^2+c^2}{d^2-c^2} \tag{3.2}$$

For $r < c$ the stresses in the plane of the annular slit are [10]

$$\sigma_z(r) = 2\sqrt{2}\, p\,(1+\operatorname{ch} x)^{3/2}\pi^{-1} q(x)$$

Let us write the following approximate expression for the function $f(x)$ from (3.2):

$$(1+\operatorname{ch} x)^{-1} = \sum_{l=0}^{N-1}\beta_l P_l(\operatorname{ch} x), \qquad 0 \leqslant x \leqslant b \tag{3.3}$$

Then the approximate value of $\sigma_z(r)$ for $b \to \infty$ $(c \to d)$ is found using the formula

$$\sigma_z(r) = 2\sqrt{2}\, p\,(1+\operatorname{ch} x)^{3/2}\pi^{-1}\sum_{l=0}^{N-1}\beta_l q_l(x) \tag{3.4}$$

where $q_l(x)$ is (3.1) with $\gamma = \pi$.

The critical tensile forces p^* at infinity, under which the slit begins to enlarge, is found from the condition [10]

$$p_* = K\pi^{-1}\lim_{r\to c}\left[\sqrt{c-r}\,\sigma_z^*(r)\right]^{-1}, \qquad \sigma_z(r) = p\sigma_z^*(r)$$

where K is the modulus of cohesion of the material. On the basis of (3.4) we have

$$p_* = \frac{K}{\sqrt{b}}\left[\left(1-\left(\frac{c}{d}\right)^4\right)\frac{d(1+\operatorname{ch} b)^3}{2c}\right]^{-1/2} \times \tag{3.5}$$
$$\left\{\sum_{l=0}^{N-1}\beta_l\frac{(2l-1)!!\,(2l+1)!!}{(l!)^2}\left(\frac{\operatorname{ch} b}{2}\right)^l\right\}^{-1}$$

which holds when the value of c is close to that of d.

Table 1 gives the values of the coefficients β_l appearing in the approximate formula (3.3) obtained by the method of least squares for $N=3$ for various values of c/d. The last column shows the relative errors of approximation given in percentage.

Table 1

c/d	β_0	β_1	β_2	%
0.3	0.893	−0.466	0.0723	0.004
0.4	0.873	−0.437	0.0638	0.03
0.5	0.846	−0.400	0.0534	0.1
0.6	0.807	−0.351	0.0413	0.5

Below we give the values of $p_*\sqrt{b}/K$ computed according to the formula (3.5) for various c/d

$c/d =$	0.4	0.5	0.6
$p_*\sqrt{b}/K =$	0.796	0.899	0.990

For comparison purposes we note that for $c/d = 0.6$, on the basis of [11], $p_*\sqrt{b}/K = 0.922$.

2°. Let us now consider the problem of torsion of a truncated sphere. The plane part of its surface adhers to a circular cylindrical punch of radius c, while the spherical part is stationary. Let d be the radius of the plane of truncation. Then the problem can be reduced (see [9], p. 390) to the dual integral equation (2.6) with

$$K(u) = u^{-1}\,\mathrm{th}\,\gamma u, \qquad f(x) = (\mathrm{ch}\,x+1)^{-1/2}\,\mathrm{th}\,\frac{x}{2}$$

$$\mathrm{ch}\,x = \frac{d^2+r^2}{d^2-r^2}, \qquad \mathrm{ch}\,b = \frac{d^2+c^2}{d^2-c^2}, \quad \gamma = \arcsin\frac{d}{R}, \quad m=1$$

where r denotes the distance from the symmetry axis in the plane of truncation, R is the radius of the sphere and $\gamma\in[0,\pi]$ characterizes the degree of trunction of the sphere. Using the relation (2.8) we can transform Eq. (2.6) to its form corresponding to $m=0$ and

$$f(x) = c_1 - 2\,(1+\mathrm{ch}\,x)^{-1/2} \tag{3.6}$$

where c_1 is a constant which shall be obtained later from the condition of integrability of the shear stresses under the punch.

The shear stresses under the punch are [9]

$$\tau_{\varphi z}(r) = -G\varepsilon\,(\mathrm{ch}\,x+1)^{3/2}\frac{d}{dx}\,q(x), \qquad \mathrm{ch}\,x = \frac{d^2+r^2}{d^2-r^2}$$

Let us approximate the function (3.6) using the expression

$$f(x) = c_1P_0(x) - \sum_{l=0}^{\infty}\beta_lP_l(x) \tag{3.7}$$

Then the approximate value of $\tau_{\varphi z}(r)$ for $b\to\infty$ $(c\to d)$ is found from the formula

$$\tau_{\varphi z}(r) = -G\varepsilon\,(\mathrm{ch}\,x+1)^{3/2}\left[c_1q_0'(x) - \sum_{l=0}^{N-1}\beta_lq_l'(x)\right] \tag{3.8}$$

where $q_l(x)$ is given by (3.1).

It can be confirmed that (3.6) has a nonintegrable singularity at $x=b$ $(r=c)$. Setting the coefficient of this singularity equal to zero, we obtain the constant c_1

$$c_1 = \Gamma\left(\frac{1}{2}+\frac{\gamma}{2\pi}\right)\Gamma^{-1}\left(1+\frac{\gamma}{2\pi}\right)\sum_{l=0}^{N-1}\beta_l\,(2l-1)!!\,(\mathrm{ch}\,b)^l\times$$

$$\times\Gamma\left[1+\frac{\gamma}{\pi}\left(l+\frac{1}{2}\right)\right]\left\{l!\,\Gamma\left[\frac{1}{2}+\frac{\gamma}{\pi}\left(l+\frac{1}{2}\right)\right]\right\}^{-1}$$

The quantity $q_l'(x)$ in this case becomes

$$q_l'(x)=\left(l+\frac{1}{2}\right)\operatorname{tg}^{-1}\left[\gamma\left(l+\frac{1}{2}\right)\right]\frac{'d}{dx}P_l(\operatorname{ch}x)+ \tag{3.9}$$
$$\sqrt{\pi}\,(2l+1)!!\operatorname{th}x\,(\operatorname{ch}b)^{l+1}\,\Gamma\left[1+\frac{\gamma}{\pi}\left(l+\frac{1}{2}\right)\right]\times$$
$$\left\{2\gamma\sqrt{\operatorname{ch}b\operatorname{ch}x}\,\Gamma\left[\frac{1}{2}+\frac{\gamma}{\pi}\left(l+\frac{1}{2}\right)\right]\right\}^{-1}\left\{\left[1-\left(\frac{\operatorname{ch}x}{\operatorname{ch}b}\right)^{\pi/\gamma}\right]^{-1/2}+\right.$$
$$\left.\frac{l\gamma|}{\pi}\sum_{n=1}^{\infty}\left[n-\frac{\gamma(2l+1)}{2\pi}\right]^{-1}\frac{(2n-1)!!}{(2\pi)!!}\left(\frac{\operatorname{ch}x}{\operatorname{ch}b}\right)^{\pi n/\gamma}\right\}$$

Table 2

c/d	β_0	β_1	β_2	%
0.3	1.92	−0.589	0.0787	0.001
0.4	1.81	−0.562	0.0707	0.01
0.5	1.88	−0.526	0.0609	0.04
0.6	1.84	−0.478	0.0490	0.1

Table 3

γ	c/d	r/c				
		0.1	0.3	0.5	0.7	0.9
3.14	0.3	0.132	0.411	0.753	1.28	2.68
3.14	0.5	0.139	0.432	0.791	1.34	2.80
3.14	0.3	0.125	0.388	0.700	1.14	2.20
3.14	0.5	0.142	0.448	0.837	1.42	2.64
0.5	0.3	0.314	0.958	1.66	2.55	4.35
0.5	0.4	0.399	1.21	2.09	3.15	5.14
0.5	0.5	0.489	1.49	2.57	3.87	6.11
1.0	0.3	0.201	0.619	1.09	1.72	3.11
1.0	0.4	0.239	0.736	1.30	2.03	3.54
1.0	0.5	0.275	0.849	1.51	2.38	4.06
1.5	0.3	0.164	0.507	0.903	1.44	2.67
1.5	0.4	0.189	0.584	1.04	1.66	2.99
1.5	0.5	0.209	0.649	1.27	1.91	3.37
2.0	0.3	0.145	0.451	0.807	1.30	2.45
2.0	0.4	0.164	0.509	0.916	1.48	2.71
2.0	0.5	0.176	0.553	1.01	1.67	3.02
2.5	0.3	0.134	0.417	0.749	1.22	2.31
2.5	0.4	0.149	0.464	0.840	1.37	2.54
2.5	0.5	0.157	0.495	0.916	1.54	2.82

The formula (3.6) for the shear stresses under the punch obviously holds when the value of c is close to that of d.

Table 2 gives the values of the coefficients β_l of the approximation (3.7) found by the method of least squares for $N = 3$ and for various values of c / d. The last column shows the relative errors of approximation given in percentage.

Table 3 gives values of $\tau_{\varphi z}(r)(G\varepsilon)^{-1}$ computed according to the formulas (3.8) and (3.9) for various values of c / d, r / c and γ. For comparison, the first two rows give the values of $\tau_{\varphi z}(r)(G\varepsilon)^{-1}$ for $\gamma = \pi$, from Table 2 of [12].

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